Rigid surface operators and S-duality: some proposals

JHEP05(2009)125

(http://iopscience.iop.org/1126-6708/2009/05/125)

Published by IOP Publishing for SISSA

Received: *January 27, 2009*
Accepted: *May 7, 2009*
Published: *May 29, 2009*

Rigid surface operators and S-duality: some proposals

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Abstract: We study surface operators in the $\mathcal{N} = 4$ supersymmetric Yang-Mills theories with gauge groups $\mathrm{SO}(n)$ and $\mathrm{Sp}(2n)$. As recently shown by Gukov and Witten these theories have a class of rigid surface operators which are expected to be related by S-duality. The rigid surface operators are of two types, unipotent and semisimple. We make explicit proposals for how the S-duality map should act on unipotent surface operators. We also discuss semisimple surface operators and make some proposals for certain subclasses of such operators.

Keywords: Supersymmetric gauge theory, Supersymmetry and Duality, Extended Supersymmetry, Duality in Gauge Field Theories

ArXiv ePrint: 0901.1833

doi:10.1088/1126-6708/2009/05/125

Contents

1 Introduction

Surface operators in gauge theories are natural generalisations of the Wilson and 't Hooft operators (which are based on curves/lines). Surface operators were almost completely overlooked for a long time. Part of the reason was that there were no clear applications of such operators as compared to the more well-known Wilson and 't Hooft operators. Recently Gukov and Witten initiated a study of surface operators [1] (see also [2] for a short review and references). Although the discussion in [1] is carried out for a specific gauge theory ($\mathcal{N}=4$ super-Yang-Mills) with a specific application in mind, surface operators are expected to be a generic feature in gauge theories.

The $\mathcal{N}=4$ supersymmetric Yang-Mills theories may well be the simplest (gauge) quantum field theories in $3+1$ dimensions. These theories have a large number of symmetries and special features. One such symmetry is the mysterious S-duality symmetry.

The S-duality conjecture [3] for the $\mathcal{N}=4$ supersymmetric four-dimensional Yang-Mills theories states that the theory with gauge group G and a value of the complexified coupling constant $\tau = \frac{\theta}{2\pi} + \frac{i}{g^2_{\mathrm{YM}}}$, where θ is the theta angle and g_{YM} is the Yang-Mills coupling constant, is equivalent to the theories arising from the transformations S and T:

$$\begin{aligned} S: \ (G,\tau) &\to (G^\vee, -1/r\tau)\,, \\ T: \ (G,\tau) &\to (G, \tau+1)\,, \end{aligned} \tag{1.1}$$

where $G^\vee$ denotes the dual group of G [4] and r is the ratio of the lengths-squared of the long and short roots of the Lie algebra of G (see e.g. [5] for a recent discussion). For the simple groups with simply-laced Lie algebras, $G^\vee$ and G are equal at the Lie algebra level. However, this is not true for all groups. Some examples of S-dual pairs, that will be studied further in this paper, are:

$\underline{G}$	$\underline{G^\vee}$	$\underline{C}$
$\mathrm{Spin}(2n{+}1)$	$\mathrm{Sp}(2n)/\mathbb{Z}_2$	$\mathbb{Z}_2$
$\mathrm{Sp}(2n)$	$\mathrm{Spin}(2n{+}1)/\mathbb{Z}_2 \equiv \mathrm{SO}(2n{+}1)$	$\mathbb{Z}_2$
$\mathrm{SO}(2n)$	$\mathrm{SO}(2n)$	$\mathbb{Z}_2\,.$

(1.2)

Here C denotes the centre of the group G.

The S-duality conjecture is well established, but has not been proven, and it is in general difficult to devise tests of the conjecture. One common strategy is to look for objects that are independent of the coupling constant and hence should have a counterpart in the dual gauge theory.

In a recent paper [6] Gukov and Witten extended their earlier analysis of surface operators and identified a subclass of surface operators in the $\mathcal{N}=4$ super-Yang-Mills theories which preserve half the supersymmetries and have the property that they are *rigid* (which essentially means that they can not be changed by an adiabatic change of τ). Rigid surface operators therefore provide a class of operators that are expected to be closed (i.e. related to each other) under S-duality. (S-duality properties of other classes of surface operators have been studied in [1, 7].)

It was shown in [6] that the rigid surface operators are of two types: unipotent and semisimple. The rigid semisimple surface operators in the theories with gauge groups $\mathrm{SO}(n)$ and $\mathrm{Sp}(2n)$ are labelled by pairs of certain partitions. Unipotent rigid surface operators arise in the limit when one of the two partitions is empty.

Partitions have also appeared in other recent works on S-duality [8, 9]. These works have in common that they count quantum states. For such states one can have quantum-mechanical state mixing which complicates the search for an S-duality map. Therefore in [8, 9] only the total number of states with certain quantum numbers were counted. The rigid surface operators on the other hand appear not to suffer from such quantum ambiguities and it therefore makes sense to look for an S-duality map, mapping a rigid surface operator in the theory with gauge group G into a rigid surface operator in the theory with gauge group $G^\vee$. In [6] the search for such an S-duality map was begun and some proposals for the S-duality map relating rigid surface operators in the B_n ($\mathrm{SO}(2n{+}1)$) and C_n ($\mathrm{Sp}(2n)$) theories for low ranks were made. A certain special subclass of unipotent rigid surface operators was also argued to be closed under S-duality. In addition, a problematic

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mismatch in the total number of rigid surface operators in the B_n and C_n theories was pointed out.

In this paper we attempt to extend the analysis begun in [6]. In particular, we make several proposals for how the S-duality map should act on certain classes of rigid surface operators in the $\mathcal{N} = 4$ B_n and C_n theories. We also make some comments and proposals for the D_n (SO($2n$)) theories.

In the next section we review the construction of rigid surface operators given in [6] and discuss some mathematical results and definitions that will be needed in later sections. We also discuss certain invariants of the surface operators, i.e. expressions that are expected to be unchanged under the S-duality map. In particular, we review the invariants proposed in [6] and also propose a new invariant, which is closely related to 'fingerprint' invariant discussed in [6]. Then in section 3 we discuss rigid surface operators in the B_n and C_n theories and make several proposals for how the S-duality map should act on certain classes of surface operators. In particular, we make a proposal for how the S-duality map should act on unipotent rigid surface operators. We also discuss semisimple surface operators and the mismatch of the total number of rigid surface operators and try to find a way to characterise the problematic surface operators. Finally, in section 4 we briefly discuss the D_n theories and make a proposal for how the S-duality map should act on unipotent rigid surface operators and also discuss a class of semisimple surface operators. In the appendix we tabulate, as an example, all rigid surface operators and their associated invariants in the SO(13) and Sp(12) theories.

2 Surface operators in $\mathcal{N} = 4$ super-Yang-Mills

The $\mathcal{N} = 4$ super-Yang-Mills theory is a four-dimensional gauge theory with gauge group G and the following field content: a gauge field (1-form), A_μ ($\mu = 0, 1, 2, 3$), four Majorana spinors ψ^a ($a = 1, 2, 3, 4$) and six real scalars, ϕ_I ($I = 1, \ldots, 6$). All fields take values in the adjoint representation of the gauge group.

Surface operators are generalisations of the Wilson and 't Hooft operators in gauge theories. Instead of being localised on a one-dimensional submanifold they are localised on a two-dimensional surface. The definition of surface operators in [1, 6] involves a generalisation of the definition of 't Hooft operators (see also [10] and references therein for a discussion of various ways to define surface operators).

A surface operator is defined by prescribing a certain singularity structure of the gauge (and scalar) fields near the surface on which the operator is supported. We only consider surface operators supported on a $\mathbb{R}^2$ submanifold (denoted D) of flat four-dimensional space. The surface D is taken to lie at $x_2 = x_3 = 0$ and the gauge 1-form in the directions normal to the surface is $A = A_2\,\mathrm{d}x^2 + A_3\,\mathrm{d}x^3$. To preserve half of the supersymmetries, the full SO(6) R symmetry group can not be unbroken. By selecting two of the six scalars in the $\mathcal{N} = 4$ super-Yang-Mills theory (ϕ_2 and ϕ_3 say) and forming $\phi = \phi_2\,\mathrm{d}x^2 + \phi_3\,\mathrm{d}x^3$,

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the conditions for preserving half of the supersymmetries can be written [6]

$$\begin{aligned} F - \phi \wedge \phi &= 0\,, \\ \mathrm{d}\phi + A \wedge \phi + \phi \wedge A &= 0\,, \\ \mathrm{d}\star\phi + A \wedge \star\phi + \star\phi \wedge A &= 0\,, \end{aligned} \tag{2.1}$$

where $F = \mathrm{d}A + A \wedge A$ as usual. The equations (2.1) are known as Hitchin's equations. A solution to these equations with a prescribed singularity along the surface D defines a surface operator.

Up to gauge transformations the most general rotation-invariant Ansatz for A and ϕ is (here $x_2 + ix_3 = re^{i\theta}$)

$$\begin{aligned} A &= a(r)\,\mathrm{d}\theta\,, \\ \phi &= c(r)\,\mathrm{d}\theta + b(r)\frac{\mathrm{d}r}{r}\,, \\ \star\phi &= -b(r)\,\mathrm{d}\theta + c(r)\frac{\mathrm{d}r}{r}\,. \end{aligned} \tag{2.2}$$

Inserting this Ansatz into (2.1) and defining $s = \ln r$ one finds that (2.1) reduce to Nahm's equations:

$$\begin{aligned} \frac{\mathrm{d}a}{\mathrm{d}s} &= [b, c]\,, \\ \frac{\mathrm{d}b}{\mathrm{d}s} &= [c, a]\,, \\ \frac{\mathrm{d}c}{\mathrm{d}s} &= [a, b]\,. \end{aligned} \tag{2.3}$$

If one is interested in conformally invariant surface operators one naively expects that scale invariance would require that a, b, c have to be independent of s (r). Nahm's equations then imply that the constant elements a, b and c need to mutually commute. Surface operators of this type were treated in [1]. The new insight in [6] was to point out another way to obtain conformally invariant surface operators.

Nahm's equations (2.3) are solved by

$$a = \frac{T_x}{s + 1/f}\,, \qquad b = \frac{T_z}{s + 1/f}\,, \qquad c = \frac{T_y}{s + 1/f}\,, \tag{2.4}$$

provided that

$$[T_x, T_y] = T_z \qquad \text{et cycl.} \tag{2.5}$$

i.e. the T_i's span a representation (in general reducible) of the su(2) Lie algebra. The T_i's also have to belong to the adjoint representation of the gauge group.

It would seem that the surface operator obtained from the solution (2.4) depends on f (for a fixed f). However, in [6] it was argued that one should think of f as being allowed to fluctuate. Then provided certain additional constraints (to be discussed below) are fulfilled, the resulting surface operator does not depend on any parameters and therefore has to be scale invariant. It is expected that it is in fact also superconformal.

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Another way to characterise the surface operators can be obtained by considering the conjugacy class (orbit under gauge conjugation) of the monodromy

$$U = P\exp\left(\oint \mathcal{A}\right)\,, \tag{2.6}$$

where $\mathcal{A} = A + i\phi$ and the integration is around a circle with constant r, near $r = 0$. Note that U belongs to the complexified gauge group and the conjugacy class is therefore a conjugacy class in the complexified gauge group. Note also that $\mathcal{F} = \mathrm{d}\mathcal{A}+\mathcal{A}\wedge\mathcal{A} = 0$, which follows from (2.1) and means that U is unchanged under deformations of the integration contour. For the solution (2.4) U becomes

$$U = P\exp\left(\frac{2\pi}{s+1/f}\,T_+\right)\,, \tag{2.7}$$

where $T_+ \equiv T_x + iT_y$ is nilpotent (strictly upper (or lower) triangular in matrix language). A conjugacy class of this type is called unipotent (the corresponding Lie algebra orbit is called nilpotent).

The above construction of surface operators does not exhaust all possibilities [6]. This can be seen by noting that there are two types of conjugacy classes in a Lie group: unipotent and semisimple. Above we only discussed unipotent classes. However, semisimple classes can also lead to rigid surface operators. The above discussion can be modified to incorporate semisimple conjugacy classes using the following construction. Consider a semisimple (diagonalisable in matrix language) element S of the gauge group and require that near the surface D,

$$S\Upsilon(r,\theta)S^{-1} = \Upsilon(r,\theta+2\pi)\,, \tag{2.8}$$

for all adjoint-valued fields Υ in the theory. This effectively breaks the gauge group to the centraliser of S (i.e. all group elements which commute with S). One can combine this with the above construction by looking for a solution to Nahm's equations which in addition also satisfies, near $r = 0$, the restriction arising from S, (2.8). At the level of conjugacy classes this combination of the two constructions means that one considers more general monodromies of the form $V = SU$, where S is semisimple and U is unipotent.

From the above discussion we see that what is needed to find the possible surface operators is a classification of unipotent and semisimple conjugacy classes. In general the construction of surface operators from conjugacy classes leads to a large variety of surface operators not all of which are expected not to depend on any parameters and to have a simple behaviour under S-duality. What is needed is a criteria which can be used to decide when a surface operator is 'rigid'.

Nilpotent orbits (unipotent conjugacy classes) have been classified by mathematicians. A nilpotent/unipotent orbit whose dimension is strictly smaller than that of any nearby orbit is called rigid. All rigid orbits have been classified (see [6] and section 7 of [12] for further details). This result will be reviewed for the classical groups in the next subsection.

There exist semisimple conjugacy classes which have the property that the centraliser (unbroken gauge group) of such a class is larger than that of any nearby class (such classes

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are called isolated in the mathematics literature, see e.g. [13], section 2). The possible isolated classes S were obtained in [6] (see also section 4.1.2 in [13]); for the classical groups, this result will be reviewed in the next subsection.

Surface operators based on monodromies of the form $V = SU$, where S is semisimple and isolated and U is unipotent and rigid will be called rigid and are expected to be superconformal and not to depend on any parameters and to have a simple behaviour under S-duality. The classification of rigid surface operators in the theories with classical gauge groups will be discussed in the next subsection.

In [6] a distinction is made between strongly rigid and weakly rigid surface operators. Throughout this paper we will only consider strongly rigid operators which we for simplicity simply refer to as rigid surface operators. The larger class including also the weakly rigid surface operators could possibly be useful in resolving some of the unsolved problems.

2.1 Some mathematical definitions and results

We saw above that rigid surface operators correspond to certain (unipotent and semisimple) conjugacy classes of the (complexified) gauge group. We summarise below the main mathematical results and definitions that will be needed in this paper. A readable mathematics reference is [12]. We will describe in detail the rigid surface operators in the theories with classical gauge groups. Since the A_n series does not have any non-trivial rigid surface operators we will concentrate on the B_n, C_n and D_n series.

It is always possible to choose a block-diagonal basis for T_+ (cf. (2.7)),

$$T_+ = \begin{pmatrix} T_+^{n_1} & & \\ & \ddots & \\ & & T_+^{n_l} \end{pmatrix}, \tag{2.9}$$

where $T_+^{n_k}$ is the 'raising' generator of the n_k-dimensional irreducible representation of su(2). For the A_n series (i.e. SU(n+1) gauge groups) the above argument gives the complete solution, but for the other classical groups, i.e. B_n (SO($2n$+1)), C_n (Sp($2n$)) and D_n (SO($2n$)), there are restrictions on the allowed dimensions of the su(2) irreps arising from the requirement that T_+ should belong to the relevant gauge group. This problem has been solved by mathematicians; see also [11] for a discussion in the Physics literature (the authors of this publication were unaware of the fact that the problem had been solved by mathematicians decades earlier). The unbroken gauge Lie algebra (i.e. the subalgebra commuting with the su(2) generators) has also been worked out.

For SO(n) (Sp($2n$)) $\sum_{k=1}^{l} n_k$ equals n ($2n$) and the restrictions on the building blocks (su(2) irreps) and unbroken Lie algebra are summarised in table 1.

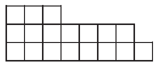
From the block-decomposition (2.9) we see see that unipotent (nilpotent) surface operators are classified by partitions. The fact that not all su(2) representations are allowed means that the classification involves restricted partitions.

A partition λ of the positive integer n is a collection of positive integers, λ_i, (the parts of the partition) such that $\sum_{i=1}^{l} \lambda_i = n$. We use the convention that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$. The integer l (the number of parts of the partition) is called the length of the partition.

Gauge group	Allowed su(2) representations	gauge enhancement
Sp($2n$)	$2m$ odd-dimensional irreps	sp($2m$)
	m even-dimensional irreps	so(m)
SO(n)	$2m$ even-dimensional irreps	sp($2m$)
	m odd-dimensional irreps	so(m)

Table 1. Restrictions on allowed representations and gauge enhancement

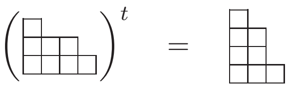
Throughout this paper we use a short-hand notation to denote partitions. For instance $3^3 2^4 1$ corresponds to $3+3+3+2+2+2+2+1$. Partitions can be added in an obvious way. If λ and κ are partitions then $\lambda+\kappa$ is the partition with parts $\lambda_i+\kappa_i$. Partitions are in a one-to-one correspondence with Young tableaux. For instance the partition $3^3 2^4 1$ corresponds to



(2.10)

An *orthogonal* partition is a partition where all even integers appear an even number of times. A *symplectic* partition is a partition for which all odd integers appear an even number of times. An orthogonal (symplectic) partition is called *rigid* if it has no gaps (i.e. $\lambda_i - \lambda_{i+1} \leq 1$ for all i) and no odd (even) integer appears exactly twice. Rigid unipotent surface operators in the B_n and D_n theories are in one-to-one correspondence with rigid orthogonal partitions of $2n+1$ and $2n$, respectively. Rigid unipotent surface operators in the C_n theories are in one-to-one correspondence with rigid symplectic partitions of $2n$. (See [6] for more details.)

The transpose of a partition is the partition obtained by interchanging the roles of the rows and columns of the Young tableau. For instance



(2.11)

The transposed partition is again a partition, but if the original partition belongs to some restricted class of partitions then the transposed partition may or may not belong to the same class.

In the theories under consideration, a partition λ is called *special* if the following condition holds

$$\begin{aligned} B_n: &\quad \lambda^t \;\text{ is orthogonal}\,, \\ C_n: &\quad \lambda^t \;\text{ is symplectic}\,, \\ D_n: &\quad \lambda^t \;\text{ is symplectic}\,. \end{aligned} \tag{2.12}$$

In particular, these definitions imply that for the B_n case all rows in the Young tableau corresponding to a rigid special partition have to be odd, whereas for the C_n and D_n cases all rows in the Young tableau corresponding to a rigid special partition have to be even.

A partition is called *rather odd* if any odd integer appears at most once.

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For the B_n , C_n and D_n theories it has been proven [6, 13] that the possible isolated semisimple conjugacy classes (cf. discussion above) correspond to diagonal matrices, S, with the only allowed elements along the diagonal being $+1$ and -1. The possible matrices S break the gauge group in the following way (at the Lie algebra level)

$$\begin{aligned} \mathrm{so}(2n{+}1) &\to \mathrm{so}(2k{+}1) \oplus \mathrm{so}(2n-2k)\,, \\ \mathrm{sp}(2n) &\to \mathrm{sp}(2k) \oplus \mathrm{sp}(2n-2k)\,, \\ \mathrm{so}(2n) &\to \mathrm{so}(2k) \oplus \mathrm{so}(2n-2k)\,. \end{aligned} \tag{2.13}$$

It then follows that the rigid semisimple surface operators in the B_n , C_n and D_n theories correspond to pairs of partitions in the following way [6]. In the B_n case a rigid semisimple surface operator is labelled by a pair of partitions $(\lambda';\lambda'')$ where λ' is a rigid B_k partition and λ'' is a rigid D_{n-k} partition. For the C_n theories a rigid semisimple surface operator is labelled by a pair of partitions $(\lambda';\lambda'')$ where λ' is a rigid C_k partition and λ'' is a rigid C_{n-k} partition (and $k \geq \lfloor \frac{n}{2} \rfloor$, where $\lfloor \cdot \rfloor$ denotes the integer part). Finally, for the D_n theories a rigid semisimple surface operator is labelled by a pair of partitions $(\lambda';\lambda'')$ where λ' is a rigid D_k partition and λ'' is a rigid D_{n-k} partition (and $k \geq \lfloor \frac{n}{2} \rfloor$). (In all the above theories, the rigid unipotent surface operators arise as a limiting case when $\lambda'' = 0$.)

The Weyl group of a simple Lie group (algebra) is a finite group of particular importance. For the Weyl group corresponding to a classical group, both its conjugacy classes[1] and unitary representations are in one-to-one correspondence with certain partitions. For the A_n case both the set of conjugacy classes and the unitary representations are in one-to-one correspondence with the set of partitions of $n{+}1$. For the B_n and C_n theories (whose Weyl groups are isomorphic) both conjugacy classes and irreducible unitary representations are in one-to one correspondence with ordered pairs of partitions $[\alpha;\beta]$ where α is a partition of n_α and β is a partition of n_β, such that $n_\alpha + n_\beta = n$. For the D_n case there is also a correspondence with pairs of partitions $[\alpha;\beta]$ where again $n_\alpha + n_\beta = n$. However, in this case there are some further refinements, but as these will not play a role in this paper we will not describe them here. Finally, we mention that even though the conjugacy classes and unitary representations are parameterised by the same set of elements there is no canonical isomorphism between the two sets (except for the A_n case).

There exist relations (maps) between the unipotent conjugacy classes (nilpotent orbits) of a simple group and its Weyl group. The Kazhdan-Lusztig map is a (in general non-bijective) map from the unipotent conjugacy classes to the set of conjugacy classes of the Weyl group. The Springer correspondence is a (injective) map from the unipotent conjugacy classes to the set of unitary representations of the Weyl group. For the classical groups these maps can be described explicitly in terms of partitions. The simplest case is A_n for which both the Kazhdan-Lusztig map and the Springer correspondence are given by the identity map.

[1] Recall that a conjugacy class, $[h]$, comprises all elements obtained from h by conjugation by a group element i.e. all elements of the form ghg^{-1}. Any element of the group belongs to precisely one conjugacy class. It is a known fact that any finite group has a certain number of conjugacy classes and an equal number of unitary representations.

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The Kazhdan-Lusztig map can be extended to the case of rigid semisimple conjugacy classes using a result due to Spaltenstein [14]. (As the Kazhdan-Lusztig map for the unipotent conjugacy classes is a special case of this construction we will not describe it separately.) Recall from the above discussion that the rigid semisimple conjugacy classes are described by pairs of partitions $(\lambda';\lambda'')$ and that the conjugacy classes of the Weyl group are described by pairs of partitions $[\alpha;\beta]$. What is needed is therefore a map between these two classes of objects. Such a map can be explicitly constructed as follows. Start by adding the two partitions: $\lambda = \lambda' + \lambda''$. Then form the symplectic partition $\mu = Sp(\lambda)$ where the function Sp is defined as follows. The parts of $\mu = Sp(\lambda)$ are given by

$$\mu_i = Sp(\lambda)_i = \begin{cases} \lambda_i + p_\lambda(i) & \text{if } \lambda_i \text{ is odd and } \lambda_i \neq \lambda_{i-p_\lambda(i)} \ , \\ \lambda_i & \text{otherwise} . \end{cases} \tag{2.14}$$

where $p_\lambda(i) = (-1)^{\sum_{k=1}^{i}\lambda_k}$. The effect of this operation is to ensure that the odd parts of the resulting partition never occur an odd number of times, i.e. the resulting partition is symplectic. As an example, if $\lambda = 7\,6^2\,5^3\,2^2\,1$ then $Sp(\lambda) = 6^4\,5^2\,2^2$.

The next step is to define the function τ from the positive integers to ± 1 in the following way. For the B_n and D_n cases $\tau(m)$ is -1 if m is even and there exists at least one μ_i such that $\mu_i = m$ and either of the following three conditions is satisfied

$$\begin{array}{ll} (i) & \mu_i \neq \lambda_i \, , \\ (ii) & \sum_{k=1}^{i}\mu_k \neq \sum_{k=1}^{i}\lambda_k \, , \\ (iii)_{\rm SO} & \lambda'_i \ \text{ is odd} . \end{array} \tag{2.15}$$

In all other instances τ is 1. For C_n the definition is the same except that condition $(iii)_{\rm SO}$ is replaced by

$$(iii)_{\rm Sp} \qquad \lambda'_i \ \text{ is even} . \tag{2.16}$$

Finally construct a pair of partitions $[\alpha;\beta]$ as follows. For each pair of parts of μ both equal to a and such that $\tau(a) = 1$ retain one part a. From the integers so obtained form the partition α. For each part of μ of size $2b$ such that $\tau(2b) = -1$ retain b. From the integers so obtained form the partition β. The resulting pair of partitions $[\alpha;\beta]$ corresponds to a conjugacy class of the Weyl group. See [14] for more details. As an example, $(\lambda';\lambda'') = (3\,2^2\,1^4;3\,2^2\,1^3)$ is mapped to $[\alpha;\beta] = [4\,;3\,1^3]$.

To describe the Springer correspondence for the classical groups it is convenient to use certain symbols introduced by Lusztig. This construction is described in section 10 of [12]. We briefly recall the main results here.

In the B_n case start by adding $l-k$ (where l is the length of the partition) to the kth part of the partition. Then split the result into two sets: one containing the even parts and one containing the odd parts. Arrange the odd parts in an increasing sequence and write them as $2f_i{+}1$ (starting with f_1). Similarly, write the even parts as $2g_i$ and arrange them in an increasing sequence (starting with g_1). Next form $\alpha_i = f_i - i + 1$ and $\beta_i = g_i - i + 1$. Note that the number of α_i's is always one more than the number of β_i's. We then write the *symbol* as

$$\begin{pmatrix} \alpha_1 & & \alpha_2 & & \alpha_3 & & \cdots \\ & \beta_1 & & \beta_2 & & \cdots & \end{pmatrix}. \tag{2.17}$$

An example illustrates the method. The B_{10} partition $\lambda = 3^3\,2^4\,1^4$ has the symbol

$$\begin{pmatrix} 0 & & 0 & & 1 & & 1 & & 1 & & 1 \\ & 1 & & 1 & & 1 & & 1 & & 2 & \end{pmatrix}. \tag{2.18}$$

Viewing the two rows of the symbol as two partitions gives the Springer correspondence, since the resulting pair of partitions corresponds to a unitary representation of the Weyl group.

For the C_n theory the symbol is formed in an analogous way. If the length of the partition is even, first append an extra 0 as the last part of the partition; if the length is odd leave the partition unchanged. Then construct f_i and g_i as in the B_n case and form $\alpha_i = g_i - i + 1$ and $\beta_i = f_i - i + 1$. The number of α_i is again one more than the number of β_i and the symbol is written as in (2.17). As an example the C_{10} partition $\lambda = 3^2\,2^6\,1^2$ has the symbol (2.18).

For the D_n theory one forms f_i and g_i exactly as in the B_n case. The difference as compared to the B_n case is that now the number of f_i and g_i are equal. This means that there are two ways to write the symbol. For reasons that will become clear we use the definition $\alpha_i = g_i - i + 1$ and $\beta_i = f_i - i + 1$, i.e. the opposite rule compared to the B_n case. Conventionally one writes the symbol with two rows of equal length. However, since we are only interested in rigid partitions which always have at least one part equal to 1 and hence $\beta_1 = 0$ we will omit this entry (and relabel $\beta_2 \to \beta_1$ etc.) when writing the symbol to ensure that the number of α_i is one more than the number of β_i just as in the B_n and C_n cases. As an example the rigid D_{10} partition $\lambda = 4^2\,3\,2^2\,1^5$ then has the symbol

$$\begin{pmatrix} 1 & & 1 & & 2 & & 2 & & 2 \\ & 0 & & 0 & & 0 & & 2 & \end{pmatrix}. \tag{2.19}$$

As mentioned above the map provided by the Springer correspondence is only injective. There exists a way to extend it to a bijection. We will not describe this extended Springer correspondence here as the relevance (if any) to surface operators is not clear.

The symbols as defined above provide an alternative characterisation of special partitions/surface operators. In the B_n and C_n theories a symbol is special if $\alpha_1 \leq \beta_1 \leq \alpha_2 + 1 \leq \beta_2 + 1 \leq \cdots$. (The rigidity restriction can also be translated into the language of symbols.) In the D_n theory a rigid symbol (defined as above) is special if $\alpha_1 \leq \beta_1 + 1 \leq \alpha_2 + 1 \leq \beta_2 + 2 \leq \cdots$.

A generalisation of the Springer correspondence to rigid semisimple conjugacy classes will be discussed in the following subsection.

2.2 Invariants of surface operators: dimension, fingerprints and symbols

To investigate how the S-duality map acts on rigid surface operators it is very helpful to find invariants of the surface operators, i.e. expressions which do not change under the S-duality map. In [6] it was pointed out that the most basic invariant of a (rigid) surface operator is the dimension, d, of the associated orbit. This quantity is calculated as follows [6, 12]:

$$\begin{aligned} B_n &: d = 2n^2 + n - \tfrac{1}{2}\sum_k (s'_k)^2 - \tfrac{1}{2}\sum_k (s''_k)^2 + \tfrac{1}{2}\sum_{k \text{ odd}} r'_k + \tfrac{1}{2}\sum_{k \text{ odd}} r''_k\,, \\ C_n &: d = 2n^2 + n - \tfrac{1}{2}\sum_k (s'_k)^2 - \tfrac{1}{2}\sum_k (s''_k)^2 - \tfrac{1}{2}\sum_{k \text{ odd}} r'_k - \tfrac{1}{2}\sum_{k \text{ odd}} r''_k\,, \\ D_n &: d = 2n^2 - n - \tfrac{1}{2}\sum_k (s'_k)^2 - \tfrac{1}{2}\sum_k (s''_k)^2 + \tfrac{1}{2}\sum_{k \text{ odd}} r'_k + \tfrac{1}{2}\sum_{k \text{ odd}} r''_k\,. \end{aligned} \tag{2.20}$$

Here s'_k denotes the number of parts of λ''s that are larger than or equal to k and r'_k denotes the number of parts of λ' that are equal to k. The definitions of s''_k and r''_k are the same with respect to λ''.

In [6] another more refined invariant was also constructed. This invariant arose by considering the singular behaviour of the fields near $r = 0$. It was shown that the mathematical description of this invariant is precisely the Weyl group conjugacy class associated with the surface operator via the Kazhdan-Lusztig map. This means that the pair of partitions $[\alpha; \beta]$ constructed from $(\lambda'; \lambda'')$ as in the previous subsection should not change under S-duality. In [6] the Weyl group conjugacy class arising from the Kazhdan-Lusztig map was referred to as the *fingerprint* of the surface operator; we will use this terminology throughout.

We now propose another invariant of rigid surface operators. This invariant is similar to the fingerprints but is based on the Springer correspondence rather than on the Kazhdan-Lusztig map.

The proposed invariant involves an extension of the Springer correspondence to rigid semisimple conjugacy classes and is constructed as follows (a similar construction appears in [13]). Calculate the symbols for both λ' and λ'' using the prescriptions given in the previous subsection and then add the two results 'from the right', i.e. write the symbols right adjusted and simply add the entries that are 'in the same place'. An example illustrates the addition rule:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 2 & 2 \\ & 1 & 2 & 2 & 2 & 2 & 3 \end{pmatrix}. \tag{2.21}$$

We refer to the resulting expression as the *symbol* of the surface operator.

It turns out that the symbol of a rigid surface operator contains the same amount of information as the fingerprint in the sense that if two rigid surface operators have the same symbols they also have the same fingerprints and vice versa. (We have not rigorously proven this statement but we have checked it in many cases.) The fact that the symbol is not an essentially new invariant is perhaps a bit disappointing but there are certain advantages of the symbols compared to the fingerprints since they are easier to calculate and their properties were quite useful in finding the S-duality maps we propose in later sections. In particular, if one want to find all possible duals of a certain (rigid) surface operator one simply looks at all possible ways of splitting the corresponding symbol into two (rigid) symbols in the dual theory. There is always only a finite number of possibilities.

2.3 Invariants of surface operators: centre and topology

In [6] further discrete invariants were also constructed. We briefly recall the definitions here. Given a surface operator corresponding to some V one can form ζV where ζ is a non-trivial element of the centre of the gauge group. If these two expressions correspond to two different surface operators then in the terminology of [6] one says that the surface operator can detect the centre. However, if one can find a group element g such that $gVg^{-1} = \zeta V$ then V and ζV belong to the same conjugacy class and do not correspond to different surface operators.

Unipotent (rigid) surface operators can always detect the centre [6]. For rigid semisimple surface operators the situation is more involved. In the B_n case we should consider the gauge group Spin($2n+1$) with centre $\mathbb{Z}_2$ generated by -1. Since both the rigid partitions λ' and λ'' have at least one part equal to 1 (which corresponds to the trivial one-dimensional su(2) representation) then in the projection to the SO($2n+1$) theory V takes the form

$$\left(\begin{array}{cc|cc} 1 & \cdots & & \\ \vdots & \ddots & & \\ \hline & & -1 & \cdots \\ & & \vdots & \ddots \end{array}\right). \tag{2.22}$$

Now this matrix lifts to $V = \gamma_2 f(\gamma_3, \ldots, \gamma_{2n})$ in the Spin($2n+1$) theory, where γ_i are the usual gamma matrices: $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$. This follows from the lifting

$$\mathrm{O}(2) \ni \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \to \gamma_2 \in \mathrm{Pin}(2)\,. \tag{2.23}$$

If we then do a gauge rotation with $g = \gamma_1\gamma_2$ we find $gVg^{-1} = -V$. This means that rigid semisimple surface operators can never detect the centre in the B_n theory.

In the C_n theory (i.e. Sp($2n$) with centre $\mathbb{Z}_2$ generated by -1), we note that if λ' and λ'' have an odd-dimensional part (or a pair of even-dimensional parts) in common, then these correspond to a block

$$\begin{pmatrix} t_+ & 0 \\ 0 & -t_+ \end{pmatrix} \equiv t_+ \otimes \sigma_z\,, \tag{2.24}$$

where t_+ belongs to a single odd-dimensional su(2) representation (or a sum of even-dimensional su(2) representations). In this case the symplectic unit acts inside t_+ and does not affect the 2×2 block structure. If we then do a gauge rotation which in the relevant sector looks like $g = \mathbb{1} \otimes \sigma_x$ we find that the above block (2.24) gets multiplied by -1. Repeating this argument we see that if $\lambda' = \lambda''$ we can find a group element such that $gVg^{-1} = -V$, which means that such surface operators can not detect the centre. However, it appears that there are additional semisimple surface operators which can not detect the centre.[2] For the above argument to go through it looks to be sufficient that the number of times a given odd-dimensional representation (or pair of even-dimensional representations) appear in the two semisimple factors are equal mod 2 (subject also to the condition that the representation(s) can not appear in only one of the two semisimple factors). Surface operators in the C_n theory which fulfill this requirement seem not to be

[2] If true, this fact will lead to some puzzles in later sections; we therefore suspect that there is a fault in the reasoning.

able to detect the centre. For instance, if an odd-dimensional irreducible representation appears three times in the first factor and once in the second we get a diagonal matrix similar to (2.24) but with t_+ appearing three times and $-t_+$ once along the diagonal. If we then perform the above gauge rotation in each of the the three 2×2 subblocks containing $-t_+$ and one of the three t_+ we find that the diagonal 4×4 matrix gets multiplied by an overall -1.

In addition to the above construction based on the centre, a related quantity was also introduced in [6]. This 'topology' quantity involves the homology groups $\pi_1(H)$ and $\pi_1(G)$ (where G is the gauge group and H is the subgroup of G left unbroken by V) rather than the centre. We will not describe the construction here (see [6] for details); instead we only give the criterion for when a surface operator can 'detect topology'. In the B_n and C_n theories the 'detects/does not detect topology' property is a $\mathbb{Z}_2$ quantum number just like the 'detects/does not detect the centre' property is.

In the B_n theory a surface operator can detect topology provided the corresponding partitions λ' and λ'' are not both rather odd [6]. In the C_n theory a surface operator can detect topology provided the corresponding partitions λ' and λ'' are both special [6]. (Note the relation with the table in Corollary 6.1.6 in [12].)

In [6] it was argued that the two discrete quantum numbers discussed above should be interchanged under S-duality, so that if a surface operator can detect topology then its dual should detect the centre and vice versa.

3 Rigid surface operators in the B_n/C_n theories

In this section we discuss the B_n and C_n theories and try to obtain information about the S-duality map between the rigid surface operators in these two theories.

3.1 Generating functions

Generating functions proved to be very useful in the works [8, 9]. The generating functions for the total number of rigid surface operators clearly contains less information than an explicit S-duality map acting on the rigid surface operators, but they could still prove to be important as a testing ground in the search for the exact map. We therefore start with a discussion of the generating functions. In the formulæ below we use the notation

$$(a, q)_k := \prod_{n=0}^{k-1} (1 - aq^n) \,. \tag{3.1}$$

The total number of rigid unipotent operators in the $\mathrm{SO}(n)$ theory is given by the coefficient in front of q^n in (the extra 1 is added for later convenience)

$$1 + \sum_{k=1}^{\infty} \left[\sum_{\substack{i_1=1 \\ i_1\neq 2}}^{\infty} q^{i_1} \sum_{i_2=1}^{\infty} q^{22i_2} \cdots \sum_{i_{2k-2}=1}^{\infty} q^{2(2k-2)i_{2k-2}} \left(\sum_{\substack{i_{2k-1}=1 \\ i_{2k-1}\neq 2}}^{\infty} q^{(2k-1)i_{2k-1}} + \sum_{i_{2k}}^{\infty} q^{2(2k)i_{2k}} \right) \right]$$
$$= 1 + \sum_{k=1}^{\infty} \frac{q^{3k^2-2k}(-q^3; q^6)_k}{(q^2; q^2)_{2k}} \equiv f(q) \,. \tag{3.2}$$

Similarly, the total number of rigid unipotent operators in the Sp$(2n)$ theory is given by the coefficient in front of q^{2n} in (again we added an extra 1)

$$\begin{aligned} &1+\sum_{k=1}^{\infty}\left[\sum_{i_1=1}^{\infty}q^{2i_1}\sum_{\substack{i_2=1\\ i_2\neq 2}}^{\infty}q^{2i_2}\cdots\sum_{\substack{i_{2k-2}=1\\ i_{2k-2}\neq 2}}^{\infty}q^{(2k-2)i_{2k-2}}\left(\sum_{i_{2k-1}=1}^{\infty}q^{2(2k-1)i_{2k-1}}+\sum_{\substack{i_{2k}=1\\ i_{2k}\neq 2}}^{\infty}q^{2ki_{2k}}\right)\right]\\ &=1+\sum_{k=1}^{\infty}\frac{q^{3k^2-k}(1-q^{4k}+q^{6k})(-q^6;q^6)_k}{(1-q^{2k}+q^{4k})(q^2;q^2)_{2k}}\equiv g(q)\,. \end{aligned} \tag{3.3}$$

Using the result (3.2), the generating function for the total number of rigid surface operators (both unipotent and semisimple) in the B_n theories becomes

$$\left[f(q)^2-f(-q)^2\right]/4\,. \tag{3.4}$$

Similarly in the C_n case we find using (3.3) the following generating function for the total number of rigid surface operators (here we multiplied the result by an extra factor of q to facilitate the comparison with the B_n result)

$$q\,\left[g(q)^2+g(q^2)\right]/2\,. \tag{3.5}$$

By expanding the above two expressions (3.4) and (3.5) one finds that the difference is

$$q^9+2\,q^{11}+4\,q^{13}+5\,q^{15}+9\,q^{17}+12\,q^{19}+17\,q^{21}+23\,q^{23}+\cdots \tag{3.6}$$

and hence there is a discrepancy between the number of rigid surface operators in the B_n and C_n theories. This discrepancy was first observed in the B_4/C_4 theories in [6]. From the above expressions one gets some further insight into the discrepancy. It appears that (for $n\geq 4$) the number of rigid surface operators is larger in the B_n theory as compared to the C_n theory and that the excess grows with the rank, n. However, the excess number of states divided by the total number appears to approach zero as $n\to\infty$. This leads to the hope that only a minor modification is needed to make the numbers match. This dovetails nicely with the fact that most rigid surface operators do seem to have candidate duals. The discrepancy is clearly a major problem but we will ignore it for now and try to identify certain subsets of rigid surface operators and make proposals for how the S-duality map should acts on these. We will return to the discrepancy issue in section 3.8.

3.2 S-duality map between rigid special unipotent surface operators [6]

In [6] it was proposed that the special rigid unipotent surface operators in the B_n and C_n theories are related by S-duality. As discussed above, special rigid unipotent surface operators in the B_n theories are characterised by Young tableaux where all the rows have an odd number of boxes and the number of rows is also odd. (The tableaux of course also satisfy the conditions required for them to be rigid.) Special rigid unipotent surface operators in the C_n theories are described by Young tableaux where all the rows have an even number of boxes (plus the rigidity conditions).

The proposed S-duality map (which we will call X_S) from the special rigid unipotent surface operators in the B_n theory to those in the C_n theory acts in the following way [6]

$$\begin{aligned} X_S: \quad & m^{2n_m+1}\,(m-1)^{2n_{m-1}}\,(m-2)^{2n_{m-2}}\cdots 2^{n_2}\,1^{2n_1} \\ \mapsto\; & m^{2n_m}\,(m-1)^{2n_{m-1}+2}\,(m-2)^{2n_{m-2}-2}\cdots 2^{n_2+2}\,1^{2n_1-2}\,. \end{aligned} \tag{3.7}$$

Here m has to be odd in order for the first object to be a B_n partition. Furthermore, it is clear that the map is a bijection so that X_S^{-1} is well defined.

The map (3.7) preserves the rigidity conditions since $n_{2j+1} \neq 1$ on the B_n side implies $n_{2j} \neq 1$ on the C_n side. Note that the map (3.7) is essentially the 'p_C collapse' described in section 6.3 in [12] or more precisely the map Sp described above and in [14]. The inverse operation, X_S^{-1}, is essentially the 'p^B expansion' also described in section 6.3 in [12].

The matching of the generating functions for the special unipotent surface operators in the B_n and C_n theories is the equality:

$$\sum_{k=1}^{\infty} \frac{q^{6k^2-8k+3}}{(q^2;q^2)_{2k-1}} = q + \sum_{k=1}^{\infty} \frac{q^{6k^2-4k+1}(1-q^{4k}+q^{8k})}{(q^2;q^2)_{2k}}\,. \tag{3.8}$$

In [6] it was checked that the fingerprints and discrete invariants are preserved by the map. On both sides the fingerprints become

$$\left[\cdots 5^{n_5-1}\,3^{n_3-1}\,1^{n_1-1}; \cdots 2^{2n_4+2}\,1^{2n_2+2}\right]. \tag{3.9}$$

On the B_n side rigid special unipotent surface operators can detect the centre and the topology. The same is true on the C_n side.

Above we proposed an alternative invariant based on symbols. This invariant can be calculated on both sides and gives:

$$\begin{pmatrix} 0\cdots 0 & \overbrace{1\cdots 1}^{n_2} & 1\ldots 1 & \cdots \\ \underbrace{1\cdots 1}_{n_1} & 1\cdots 1 & \underbrace{2\cdots 2}_{n_3} & \cdots \end{pmatrix}. \tag{3.10}$$

Note that the jumps in the entries occur on different rows each time. This alternating behaviour is characteristic of unipotent special surface operators.

It is not entirely obvious that the S-duality map (3.7) is uniquely fixed by the requirement that it preserves the invariants. Nevertheless, it is a simple rule and we will assume that it is the correct map.

3.3 S-duality map for rigid rather odd unipotent surface operators

Above we saw that the special unipotent operators are related by S-duality. In this subsection we will discuss another subclass of operators in the B_n theories and identify their duals. This subclass consists of all B_n operators for which one can detect the centre but not the topology. From the discussion in section 2.3 we find that surface operators with these properties are rigid rather odd unipotent surface operators. Such surface operators

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correspond to partitions of the form $\cdots 5\, 4^{2n_4}\, 3\, 2^{2n_2}\, 1$ (note that the number of odd integers has to be odd for the surface operator to belong to B_n).

We propose the duality map

$$\begin{aligned} &(\cdots 9\, 8^{2n_8}\, 7\, 6^{2n_6} 5\, 4^{2n_4}\, 3\, 2^{2n_2}\, 1\, ; \emptyset) \\ \mapsto\ &(\cdots 4^{2n_8+2} 3^{2n_6} 2^{2n_4+2} 1^{2n_2}\, ; \cdots 4^{2n_8+2} 3^{2n_6} 2^{2n_4+2} 1^{2n_2})\,. \end{aligned} \tag{3.11}$$

We first note that the proposed duals are rigid (including the constraint that even parts can not appear with multiplicity 2. Furthermore, the duals are special semisimple surface operators constructed out of two equal partitions. Since the surface operators are special they can detect the topology and since $\lambda' = \lambda''$ they can not detect the centre as required (cf. section 2.3). Next one can easily calculate the fingerprints on both sides to obtain

$$\left[\cdots 6^{n_6}\, 2^{n_2}\, ; \cdots 4^{2n_8+2} 2^{2n_4+2}\right]\,. \tag{3.12}$$

The matching of symbols can also be checked:

$$\begin{aligned} &\begin{pmatrix} 0 & 0\cdots 0 & \overbrace{2\cdots 2}^{n_4+1} & 2\ldots 2 & \cdots \\ & \underbrace{2\cdots 2}_{n_2} & 2\cdots 2 & \underbrace{4\cdots 4}_{n_6} & \cdots \end{pmatrix} = \\ &\begin{pmatrix} 0 & 0\cdots 0 & \overbrace{1\cdots 1}^{n_4+1} & 1\ldots 1 & \cdots \\ & \underbrace{1\cdots 1}_{n_2} & 1\cdots 1 & \underbrace{2\cdots 2}_{n_6} & \cdots \end{pmatrix} + \begin{pmatrix} 0 & 0\cdots 0 & \overbrace{1\cdots 1}^{n_4+1} & 1\ldots 1 & \cdots \\ & \underbrace{1\cdots 1}_{n_2} & 1\cdots 1 & \underbrace{2\cdots 2}_{n_6} & \cdots \end{pmatrix}. \end{aligned} \tag{3.13}$$

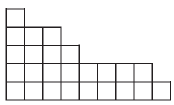
Thus the proposed dual pair passes all consistency checks that we know of.

The check of the matching of symbols is particularly revealing. This is because the symbol on the B_n side only involves even numbers, and jumps alternate between the two rows. There is only *one* way to write it as a sum of two rigid special C_n symbols (recall from the above discussion, cf. (3.10), that rigid special C_n symbols also have jumps alternating between the two rows but the jumps only involve a difference of $+1$ each time). The fact that the surface operators need to detect topology on the C_n side (since the centre can be detected on the B_n side), requires the C_n partitions to be special and we can be confident that we have found the right dual. For this reason the class of rigid rather odd unipotent operators is in a sense even simpler that the class of special unipotent operators whose duals where identified in [6] and described above.

The matching of the generating functions of the two dual classes is the equality:

$$\sum_{k=1}^{\infty} \frac{q^{3k^2-2k}}{(q^4;q^4)_k} = q + \sum_{k=1}^{\infty} \frac{q^{12k^2-8k+1}(1-q^{8k}+q^{16k})}{(q^4;q^4)_{2k}}\,. \tag{3.14}$$

Let us now describe how the map (3.11) acts on the partitions in a way which will facilitate the generalisation to all rigid unipotent B_n surface operators (not necessarily special or rather odd). For simplicity we focus on the case $5\, 4^2\, 3\, 2^4\, 1$. The Young tableau is



(3.15)

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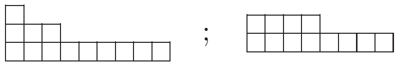
Let us split this tableau into one tableau constructed from the rows with an odd number of boxes and one tableau constructed from the rows with an even number of boxes; we get



(3.16)

Next we apply the map (3.7) to the tableau constructed from the odd rows to obtain

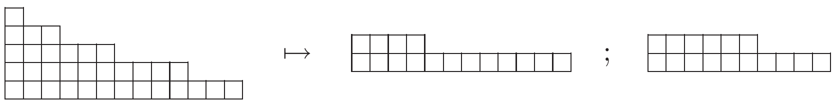
(3.17)

Thus we arrive at the rigid surface operator $(2^4\,1^4; 2^4\,1^4)$ which agrees with the dual proposed above. The general prescription is clear: split the Young tableau into one tableau constructed from the rows with an odd number of boxes and one constructed from the rows with an even number of boxes, and then apply the map (3.7) to the tableau with odd rows. The two tableaux so constructed correspond the two (equal) rigid special symplectic partitions given in (3.11).

3.4 A proposal for the S-duality map for rigid unipotent operators

From the discussion in the previous two subsections, a natural generalisation of the proposed S-duality map to all unipotent surface operators in the B_n theories now presents itself: we simply apply the same manipulation rule at the level of Young tableaux that we did at the end of the previous subsection. First note that this algorithm always gives a rigid special semisimple surface operator in the C_n theory: The first tableaux (the one with only odd rows) is always a special rigid partition in some B_k theory and the map (3.7) turns this into a special partition in the C_k theory. The second partition (the one with only even rows) already corresponds to a special rigid partition in the C_{n-k} theory and is left untouched. The fact that we obtain a special C_n surface operator shows that it can detect topology on the C_n side which is consistent with the fact that the unipotent B_n surface operators can detect the centre. Also note that if the B_n tableaux is special (i.e. has only odd rows) we recover the map proposed in [6].

As an example of the procedure, consider the unipotent B_{16} operator corresponding to the partition $\lambda = 5\,4^2\,3^3\,2^4\,1^3$. Applying the proposed map we find



(3.18)

i.e. $(2^4\,1^8\,,\,2^6\,1^4)$. This semisimple C_{16} operator has identical fingerprint and symbol to the B_{16} operator we started with.

To check that the proposed map preserves the symbols is not too difficult. This is because the Young tableau operations we have performed have direct counterparts in the symbols. Splitting a rigid Young tableau along rows corresponds to splitting the symbol at the places where the values of the entries jump. In particular, splitting the Young tableau into sets of even and odd rows corresponds to splitting the symbol into two special symbols. To see this first note that there can be at most two consecutive jumps within either of the rows of the symbol before the jump has to switch rows (this follows from the orthogonal/symplectic constraint). The splitting into even and odd rows is at points where the second of two such consequtive jumps occurs. This is best illustrated by an example. In the above rank 16 tableau (3.18) the splitting of symbols is as in (2.21).

It is also possible to show that the proposed map preserves the fingerprints. This is a little more involved. The first thing to note is that on the B_n side $\lambda = \lambda_{\text{even}} + \lambda_{\text{odd}}$ and $\mu = Sp(\lambda) = Sp(\lambda_{\text{odd}}) + \lambda_{\text{even}}$. This result follows from the definition (2.14). Note that the longest row in a rigid B_n partition always contains an odd number of boxes. The following two rows are either both of odd length or both of even length. This pairwise pattern then continues. If the tableau has an even number of rows the row of shortest length has to be even.

On the C_n side $\lambda' = X_S(\lambda_{\text{odd}}) = Sp(\lambda_{\text{odd}})$ and $\lambda'' = \lambda_{\text{even}}$ which implies that $\mu = \lambda$ since both λ' and λ'' are special C_n partitions which means that so is $\lambda \equiv \lambda' + \lambda''$. Since $\mu = \lambda$ it then follows from the definition of the map τ that τ is -1 only when μ_i is even and λ'_i is even, i.e. when both λ'_i and λ''_i are even.

We need to show that τ is also -1 for the same μ_i on the B_n side. When μ_i is even, either both of the corresponding parts of $Sp(\lambda_{\text{odd}})$ and λ_{even} are odd or they are even. If both are odd we see from (3.7) that the first two conditions in (2.15) are fulfilled (the third condition is moot when $\lambda'' = \emptyset$). Hence $\tau = +1$ for such μ_i. If both are even it follows from (3.7) that for the even parts of $Sp(\lambda_{\text{odd}})$ at least one of the corresponding parts of λ_{odd} is different. This implies that we have $\tau = -1$. This is the same result as on the C_n side, hence the fingerprints are the same.

As already mentioned the fact that the B_n unipotent surface operators detect the centre is consistent with the fact that the proposed duals detect topology. The B_n unipotent surface operators that detect topology (i.e. the ones that are not rather odd) should have duals which detect the centre. Here we encounter a puzzle: from the discussion in section 2.3 it seems that some special rigid semisimple C_n operators with $\lambda' \neq \lambda''$ do not detect the centre. If so, this would be problematic for our proposed map. This leads us to suspect, as was already mentioned in footnote 2, that the arguments in section 2.3 are not completely correct. On the other hand, if the arguments are correct then we have a more severe problem since there are in many cases no other possible duals apart from the ones arising via our proposed map (for instance, this is the case for the surface operators with orbit-dimension 20 in the rank 6 example listed in the appendix). Another puzzling aspect of a similar nature is the following. In our proposal, the unipotent rigid B_n surface operators get mapped into special rigid semisimple C_n surface operators. But, the number of special rigid semisimple surface operators in the C_n theories is larger than the number of rigid unipotent surface operators in the B_n theories. This is problematic since we argued in section 2.3 that the special C_n surface operators detect topology whereas the only B_n surface operators which detect the centre are the unipotent ones. On the other hand based only on their fingerprints/symbols the extra rigid special C_n surface operators appear to have candidate rigid B_n duals.

Turning to the unipotent operators on the C_n side we can make a similar proposal for the dual of these operators. Starting with the Young tableau corresponding to a rigid unipotent C_n operator we split it into even-row and odd-row tableaux as in the B_n case.

(Note that the number of rows in the odd-row tableau is always even.) We then apply the map X_S^{-1} to the even-row tableau to obtain a B_k tableau (this guarantees that we reproduce the map in [6] for the special C_n operators). From the odd-row tableaux we want to obtain a rigid D_{n-k} partition (since the operation on the even-row tableau gave us a B_k partition). To accomplish this goal, we apply the following map

$$\begin{aligned} Y_S: \quad & m^{2n_m+1}\,(m-1)^{2n_{m-1}}\,(m-2)^{2n_{m-2}}\cdots 2^{n_2}\,1^{2n_1} \\ \mapsto\ & m^{2n_m}\,(m-1)^{2n_{m-1}+2}\,(m-2)^{2n_{m-2}-2}\cdots 2^{n_2-2}\,1^{2n_1+2}\,. \end{aligned} \tag{3.19}$$

Here m has to be even in order for the first element to be a C_k partition. This map is very similar to the map (3.7) and takes a special C_k partition to a special D_k partition (note that the map preserves the number of boxes). The map (3.19) is simply the 'p_D collapse' mentioned in section 6.3 in [12]. The inverse map, Y_S^{-1}, is the 'p^C expansion' (cf. section 6.3 in [12]).

The proposed map therefore takes us from a rigid unipotent C_n surface operator to a rigid special semisimple surface operator in the B_n theory. Since such a surface operator on the B_n side is never rather odd we can detect topology on the B_n side which matches the fact that we can detect the centre on the C_n side. Furthermore, the map maps unipotent C_n surface operators which can also detect topology (the special unipotent surface operators) into B_n surface operators which can also detect the centre (special unipotent surface operators). The proposed S-duality map for unipotent C_n surface operators therefore does not suffer from the problems mentioned above for the map of B_n unipotent surface operators, however, in the present case the map is less unique since there is no clear reason why the semisimple B_n duals should be special.

Again one can check that the symbols match for the proposed dual pairs. The method is completely analogous to the one used for the unipotent B_n surface operators so we will not repeat the details.

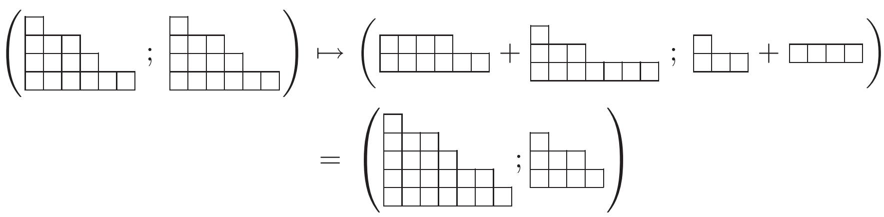
To verify that the fingerprints also match we first note that the longest two rows in a rigid C_n partition both contain either an odd number or an even number of boxes. This pairwise pattern then continues. If the tableau has an odd number of rows the row of shortest length has to contain an even number of boxes. Since the unipotent C_n partition is symplectic we have $\mu = Sp(\lambda) = \lambda$. It follows from this result that τ is -1 for all even μ_i. From the above properties of rigid C_n partitions, it also follows that the corresponding $\lambda_{\text{even},i}$ and $\lambda_{\text{odd},i}$ both have to be even. On the B_n side we have $\lambda = X_S^{-1}\lambda_{\text{even}} + Y_S\lambda_{\text{odd}}$ and $\mu = \lambda_{\text{even}} + \lambda_{\text{odd}}$ (which follows from the definitions of X_S and Y_S). As above, when μ_i is even we have that the corresponding $\lambda_{\text{even},i}$ and $\lambda_{\text{odd},i}$ both have to be even. When $\lambda_{\text{even},i}$ is even there exists an i such that μ_i and λ_i differ, which means that τ is -1 for such i. This agrees with the C_n result and the fingerprints are therefore the same.

We close this section by pointing out that in [15], section 13.3, Lusztig constructs a map from unipotent (not necessarily rigid) B_n $[C_n]$ conjugacy classes to special (not necessarily rigid) C_n $[B_n]$ semisimple conjugacy classes. The map is not described in a very explicit way. However, in a later work [13] a much more explicit map is constructed. The maps constructed in section 4.2 of [13] are very similar to the maps we have proposed. But, somewhat surprisingly, they are not the same maps since the maps in [13], as far as we can see, do not preserve the rigidity conditions.

3.5 A proposal for the S-duality map for $(\rho\,;\rho)$ C_n surface operators

Semisimple surface operators in the C_n theories for which λ' and λ'' are equal can not detect the centre (see section 2.3 above). Note that n has to be even in order for such surface operators/partitions to exist. We argued above that the $(\rho\,;\rho)$ C_n surface operators which are also special are dual to the rigid rather odd unipotent surface operators in the B_n theory. We will now make a proposal for the dual of a general rigid $(\rho\,;\rho)$ C_n surface operator. Since such surface operators can detect neither centre nor topology one expects the dual to be given by rigid rather odd semisimple operators in the B_n theory since such operators have the same properties.

Start by splitting the two equal tableaux into even-row and odd-row tableaux as above. Next apply the map (3.19) to one of the odd-row tableaux and apply the inverse of (3.7) to the even-row tableau in the other semisimple factor. Then add the altered and unaltered even-row tableaux to form one of the two partitions in a semisimple B_n operator. Finally, do the same to the odd-row tableaux. In other words, the resulting B_n partition becomes $(\rho_{\text{even}} + X_S^{-1}\rho_{\text{even}}\,; \rho_{\text{odd}} + Y_S\rho_{\text{odd}})$. Note that the first partition is a B_k partition and the second factor is a D_{n-k} partition. As an example consider the C_{14} operator $(4\,3^2\,2\,1^2\,;\,4\,3^2\,2\,1^2)$. Applying the suggested map we find:



(3.20)

i.e. the semisimple B_{14} operator $(5\,4^2\,3\,2^2\,1\,;\,3\,2^2\,1)$ which is rather odd as expected. Note that if the even-row tableaux ρ_{even} is empty the inverse map (3.7) applied to it gives the partition 1.

To check that the symbols match one can use the same methods as in previous cases. The C_n symbol corresponding to $\lambda = \rho + \rho$ has entries with only even numbers. This can be split into two symbols corresponding to rather odd symbols (for which the jumps are with steps of $+2$ and alternating between the rows, cf. section 3.3) using the same methods as in section 3.4 except that now all entries are even.

To verify that the fingerprints agree we start on the C_n side where $\lambda = \rho + \rho$ and $\mu = \lambda$ since λ is symplectic (it has only even parts). From this result it follows that τ is -1 whenever ρ_i is even.

On the B_n side we have $\lambda = \rho + Y_S\rho_{\text{odd}} + X_S^{-1}\rho_{\text{even}}$ and $\mu = \rho + \rho$. When ρ_i is even there exists μ_i which differ from the corresponding λ_i and therefore τ is -1 for such μ_i. When ρ_i is odd one instead finds that τ is 1. These results agree with the ones on the C_n side and hence the fingerprints agree.

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Thus the proposed dual pairs passes all consistency checks. However, we note that the number of rather odd semisimple B_n surface operators is larger than the number of $(\rho\,;\rho)$ non-special surface operators on the C_n side.

3.6 A proposal for the S-duality map for $(1;\delta)$ B_n surface operators

Another class of surface operators for which a natural S-duality action exists are the rigid semisimple B_n surface operators that are of the form $(1;\delta)$, i.e. λ' is a B_0 partition (1) and λ'' is a D_n partition (δ). The proposed map is similar to the above examples: split the partition δ into even and odd rows and leave the odd-row tableau unchanged and apply Y_S^{-1} to the even-row tableau. Form a semisimple C_n surface operator from the resulting two partitions. This operation gives a semisimple C_n operator where both of the two partitions have only odd rows.

Note that the above map is consistent with the proposed map for unipotent C_n surface operators (when δ is special and the dual unipotent C_n operator has only odd rows) as well as with the map for $(\rho\,;\rho)$ C_n surface operators (when δ is rather odd the dual has a ρ with only odd rows).

The methods used to check that the dual pairs have the same fingerprints are similar to the previous cases. Note that the longest row in a rigid D_n partition always contains an even number of boxes. The following two rows are either both of odd length or both of even length. This pairwise pattern then continues. If the tableau has an even number of rows the row of shortest length has to be even. On the B_n side $\lambda = 1 + \delta_{\rm odd} + \delta_{\rm even}$ and $\mu = \delta_{\rm odd} + Y_S^{-1}\delta_{\rm even}$. On the C_n side $\lambda \equiv \lambda' + \lambda'' = \delta_{\rm odd} + Y_S^{-1}\delta_{\rm even}$ and $\mu = \lambda$. This implies that whenever $\delta_{{\rm odd},i}$ is even τ is -1. This can be seen to be in agreement with the B_n result (using the properties of the Y_S map).

Excluding the case when δ is rather odd, the fact that the surface operators on the B_n side can detect topology means that on the C_n side the dual surface operators should detect the centre. Although this is generically the case, it seems that if the analysis in section 2.3 is correct some of the possible duals might not detect the centre. But as already mentioned in section 3.4 and footnote 2 we suspect that there are probably some misconceptions in that analysis.

3.7 General semisimple operators: search for an S-duality map

Above we have made some proposals for how the S-duality map should act on certain subclasses of rigid surface operators. Our proposals include all unipotent rigid surface operators as well as certain subclasses of rigid semisimple operators. The goal is of course to extend the analysis to arbitrary rigid semisimple operators. However, it seems that before such an extension can be found, the reason for the mismatch of the total number of rigid surface operators in two theories must be resolved. We therefore make some preliminary comments about the rigid surface operators responsible for the mismatch in the next subsection.

3.8 Characterising the operators which seemingly have no dual

We saw in section 3.1 that there is an excess of rigid surface operators in the B_n theories (when $n \geq 4$). One could speculate that it is only the excess surface operators which are problematic and which do not have duals, but this naive guess is not correct as we will see below.

We will only attempt a preliminary analysis of which of the surface operators are problematic; our motivation is that a more thorough understanding of which surface operator do not have candidate duals might lead to progress.

Our analysis will be based on the assumption that the symbols as defined in section 2.2 are invariants and we therefore start by recalling some pertinent facts. For rigid partitions of the form $\cdots 2^{2n_2}1$ in the B_n theories the symbols take the form

$$\begin{pmatrix} 2 & & \cdots & \\ & 0 & & \cdots \end{pmatrix}, \tag{3.21}$$

whereas for rigid partitions of the form $\cdots 1^{n_1}$ with $n_1 \geq 3$ the symbols take the form

$$\begin{pmatrix} 1 & & \cdots & \\ & 0 & & \cdots \end{pmatrix}. \tag{3.22}$$

Similarly in the D_n theories one finds that the symbols take the forms

$$\begin{pmatrix} 0 & & \cdots & \\ & 2 & & \cdots \end{pmatrix}, \qquad \begin{pmatrix} 0 & & \cdots & \\ & 1 & & \cdots \end{pmatrix}, \tag{3.23}$$

for rigid partitions of the form $\cdots 2^{2n_2}1$ and $\cdots 1^{n_1}$ with $n_1 \geq 3$, respectively. In the C_n theories the symbols take the form

$$\begin{pmatrix} 1 & & \cdots & \\ & 0 & & \cdots \end{pmatrix}, \qquad \begin{pmatrix} 0 & & \cdots & \\ & 1 & & \cdots \end{pmatrix}, \tag{3.24}$$

for rigid partitions of odd and even length, respectively.

Now consider semisimple surface operator in the B_n theory with symbols

$$\begin{pmatrix} 2 & & \cdots & \\ & 2 & & \cdots \end{pmatrix}, \qquad \begin{pmatrix} 2 & & \cdots & \\ & 1 & & \cdots \end{pmatrix}, \qquad \begin{pmatrix} 1 & & \cdots & \\ & 2 & & \cdots \end{pmatrix}. \tag{3.25}$$

Surface operators with such symbols can *not* have (rigid) C_n duals since in the C_n theory such symbols can not be constructed from the sum of two symbols of the form (3.24). The above classes of B_n operators (3.25) correspond to pairs of partitions (λ', λ'') where the length of λ' is equal to the length of λ'' plus one, and one (or both) of λ' and λ'' is of the form $\cdots 2^{2n_2}1$.

There are further infinite classes of surface operators that can not have duals, e.g. the B_n ones that have symbols of the form

$$\begin{pmatrix} 1 & & 2 \cdots & \\ & 1 & & \cdots \end{pmatrix}, \qquad \begin{pmatrix} 0 & & 2 \cdots & \\ & 1 & & \cdots \end{pmatrix}, \qquad \begin{pmatrix} 0 & & 1 & \cdots \\ & 1 & & 2 \cdots \end{pmatrix}. \tag{3.26}$$

We will not attempt to classify all symbols which can appear on the B_n side but not on the C_n side. Such a classification would anyway not be the end of the story since in addition to such symbols there are also symbols which can arise from two surface operators on the B_n side but only from one on the C_n side. This is a mismatch of a different type. Examples of such symbols include

$$\begin{pmatrix} 1 & & 1 & & 1 & & \cdots \\ & 1 & & 2 & & \cdots & \end{pmatrix}, \qquad \begin{pmatrix} 1 & & 1 & & 1 & & 1 & & \cdots \\ & 1 & & 1 & & 2 & & \cdots & \end{pmatrix}. \tag{3.27}$$

The above examples have all been cases where there are too many B_n surface operators of a certain type. Based on the generating functions a natural guess would have been that this would be the only type of problem. However, perhaps somewhat surprisingly, this is not true. Starting at rank 10 states appear in the C_n theories which based on their symbols (and fingerprints) can not have duals in the B_n theories. The first example in this series is

$$(2^4\,1^2\,;3^2\,2\,1^4)\ , \tag{3.28}$$

with symbol

$$\begin{pmatrix} 0 & & 1 & & 2 & & 2 \\ & 1 & & 1 & & 3 & \end{pmatrix}. \tag{3.29}$$

Note that there appears to be a relation between (3.28) and the excess problematic B_4 surface operator found in [6], namely $(1^4\,;2^2\,1)$: the Young tableaux of this surface operator are obtained by removing the first rows in the two tableaux corresponding to the partitions in (3.28).

4 The D_n theories

In this section we will very briefly discuss the extension of some of the techniques used in the B_n/C_n theories to the D_n (i.e. $\mathrm{SO}(2n)$) theories. The discrete invariants are potentially more restrictive since in this case the centre of $\mathrm{Spin}(2n)$ is of order 4, but we will not make use of them here.

For unipotent D_n operators we propose that the S-dual surface operator is obtained by splitting the corresponding tableau into even- and odd-row tableaux, applying the map Y_S to the odd-row tableau (which corresponds to a C_k partition) and leaving the even-row tableau unchanged. This operation results in a special semisimple rigid D_n surface operator. One can check that the fingerprints and symbols are preserved by the map but we refrain from giving the details here.

As another example consider semisimple rigid D_n surface operators of the form $(\rho\,;\rho)$. We propose the following S-duality map. Split each ρ into even- and odd-row tableaux and apply Y_S to one of the odd-row tableau and Y_S^{-1} to one of the even-row tableau. Then add the unchanged even-row tableau and the transformed even-row tableau and do the the same for the odd-row tableau. This procedure results in a rigid semisimple rather odd D_n surface operator. Note that if ρ is rather odd from the beginning then the proposed map leaves the surface operator unchanged. Again one can check that the fingerprints and symbols are preserved by the proposed map.

5 Summary and open problems

In this paper we have made some proposals for how the S-duality map should act on certain classes of rigid surface operators in the B_n, C_n and D_n theories. In particular, we have made proposals for all unipotent rigid surface operators as well as for some classes of rigid semisimple surface operators. Our proposed maps are speculative but their descriptions are quite simple and uniform. Attemts to continuing the analysis to more general classes of semisimple surface operators are hampered by the mismatch in the total number of rigid surface operators in the B_n and C_n theories. Since the D_n theories are self-dual they might prove to be easier to study. We took some tentative steps towards a classification of the B_n/C_n rigid surface operators which can not have a dual, but the physical reason for the mismatch is still unknown. Maybe the weakly rigid surface operators discussed in [6] will play a role in the resolution. Clearly more work is required; hopefully our constructions will be helpful in making further progress.

Acknowledgments

The author wishes to thank Måns Henningson for discussions. This work was supported by a grant from the Swedish Research Council.

A Rigid surface operators in the SO(13) and Sp(12) theories

Below we list (with no particular ordering) all rigid surface operators in the Sp(12) and SO(13) theories. These tables illustrate the results in this paper. The first column lists the pair of partitions corresponding to the surface operator, the second column the dimension, the third the symbol, and the fourth the fingerprint.

$(1^{12};\emptyset)$	0	$\binom{0\ 0\ 0\ 0\ 0\ 0\ 0}{1\ 1\ 1\ 1\ 1\ 1}$	$[1^6;\emptyset]$
$(2\,1^{10};\emptyset)$	12	$\binom{1\ 1\ 1\ 1\ 1\ 1}{0\ 0\ 0\ 0\ 0}$	$[1^5;1]$
$(1^{10};1^2)$	20	$\binom{0\ 0\ 0\ 0\ 0\ 0}{1\ 1\ 1\ 1\ 2}$	$[2\,1^4;\emptyset]$
$(2^3\,1^6;\emptyset)$	30	$\binom{1\ 1\ 1\ 1\ 1}{0\ 0\ 0\ 1}$	$[1^3;1^3]$
$(2\,1^8;1^2)$	30	$\binom{1\ 1\ 1\ 1\ 1}{0\ 0\ 0\ 1}$	$[1^3;1^3]$
$(1^8;1^4)$	32	$\binom{0\ 0\ 0\ 0\ 0}{1\ 1\ 2\ 2}$	$[2^2\,1^2;\emptyset]$
$(2^4\,1^4;\emptyset)$	36	$\binom{0\ 0\ 0\ 1\ 1}{1\ 1\ 1\ 1}$	$[1^2;1^4]$
$(1^8;2\,1^2)$	36	$\binom{0\ 0\ 0\ 1\ 1}{1\ 1\ 1\ 1}$	$[1^2;1^4]$
$(1^6;1^6)$	36	$\binom{0\ 0\ 0\ 0}{2\ 2\ 2}$	$[2^3;\emptyset]$
$(2^5\,1^2;\emptyset)$	40	$\binom{1\ 1\ 1\ 1}{0\ 1\ 1}$	$[1;1^5]$

$$
\begin{array}{llcl}
(2\,1^6\,;1^4) & 40 & \begin{pmatrix}1&1&1&1\\ &0&1&1\end{pmatrix} & [1\,;1^5] \\
(1^6\,;2\,1^4) & 42 & \begin{pmatrix}0&1&1&1\\ &1&1&1\end{pmatrix} & [\emptyset\,;1^6] \\
(3^2\,2\,1^4\,;\emptyset) & 44 & \begin{pmatrix}1&1&1&1\\ &0&0&2\end{pmatrix} & [3\,1^2\,;1] \\
(2^3\,1^4\,;1^2) & 44 & \begin{pmatrix}1&1&1&1\\ &0&0&2\end{pmatrix} & [3\,1^2\,;1] \\
(2\,1^6\,;2\,1^2) & 44 & \begin{pmatrix}1&1&2&2\\ &0&0&0\end{pmatrix} & [2\,1^2\,;2] \\
(2^4\,1^2\,;1^2) & 48 & \begin{pmatrix}0&0&1&1\\ &1&1&2\end{pmatrix} & [3\,1\,;1^2] \\
(2\,1^4\,;2\,1^4) & 48 & \begin{pmatrix}2&2&2\\ &0&0\end{pmatrix} & [2^2\,;2] \\
(2^3\,1^2\,;1^4) & 50 & \begin{pmatrix}1&1&1\\ &1&2\end{pmatrix} & [3\,;1^3] \\
(2^3\,1^2\,;2\,1^2) & 54 & \begin{pmatrix}1&2&2\\ &0&1\end{pmatrix} & [3\,1\,;2] \\
(3^2\,2\,1^2\,;1^2) & 54 & \begin{pmatrix}1&1&1\\ &0&3\end{pmatrix} & [4\,1\,;1] \\
(1^{13};\emptyset) & 0 & \begin{pmatrix}0&0&0&0&0&0&0\\ &1&1&1&1&1&1\end{pmatrix} & [1^6;\emptyset] \\
(1;1^{12}) & 12 & \begin{pmatrix}1&1&1&1&1&1\\ &0&0&0&0&0\end{pmatrix} & [1^5;1] \\
(2^2\,1^9;\emptyset) & 20 & \begin{pmatrix}0&0&0&0&0&0\\ &1&1&1&1&2\end{pmatrix} & [2\,1^4;\emptyset] \\
(1;2^2\,1^8) & 30 & \begin{pmatrix}1&1&1&1&1\\ &0&0&0&1\end{pmatrix} & [1^3;1^3] \\
(1^3;1^{10}) & 30 & \begin{pmatrix}1&1&1&1&1\\ &0&0&0&1\end{pmatrix} & [1^3;1^3] \\
(2^4\,1^5;\emptyset) & 32 & \begin{pmatrix}0&0&0&0&0\\ &1&1&2&2\end{pmatrix} & [2^2\,1^2;\emptyset] \\
(3\,2^2\,1^6;\emptyset) & 36 & \begin{pmatrix}0&0&0&1&1\\ &1&1&1&1\end{pmatrix} & [1^2;1^4] \\
(1^9,1^4) & 36 & \begin{pmatrix}0&0&0&1&1\\ &1&1&1&1\end{pmatrix} & [1^2;1^4] \\
(2^6\,1;\emptyset) & 36 & \begin{pmatrix}0&0&0&0\\ &2&2&2\end{pmatrix} & [2^3;\emptyset] \\
(1;2^4\,1^4) & 40 & \begin{pmatrix}1&1&1&1\\ &0&1&1\end{pmatrix} & [1;1^5] \\
(1^5;1^8) & 40 & \begin{pmatrix}1&1&1&1\\ &0&1&1\end{pmatrix} & [1;1^5] \\
(1^7;1^6) & 42 & \begin{pmatrix}0&1&1&1\\ &1&1&1\end{pmatrix} & [\emptyset;1^6] \\
(1^3;2\,1^7) & 44 & \begin{pmatrix}1&1&1&1\\ &0&0&2\end{pmatrix} & [3\,1^2;1] \\
(2^2\,1;1^8) & 44 & \begin{pmatrix}1&1&1&1\\ &0&0&2\end{pmatrix} & [3\,1^2;1] \\
(1;3\,2^2\,1^5) & 44 & \begin{pmatrix}1&1&2&2\\ &0&0&0\end{pmatrix} & [2\,1^2;2]
\end{array}
\qquad \text{(A.1)}
$$

$(2^2 1^5; 1^4)$	48	$\begin{pmatrix} 0 & 0 & 1 & 1 \\ & 1 & 1 & 2 \end{pmatrix}$	$[3\,1; 1^2]$
$(1; 3\,2^4\,1)$	48	$\begin{pmatrix} 2 & 2 & 2 \\ & 0 & 0 \end{pmatrix}$	$[2^2; 2]$
$(2^2 1^3; 1^6)$	50	$\begin{pmatrix} 1 & 1 & 1 \\ & 1 & 2 \end{pmatrix}$	$[3; 1^3]$
$(1^5; 2^2\,1^4)$	50	$\begin{pmatrix} 1 & 1 & 1 \\ & 1 & 2 \end{pmatrix}$	$[3; 1^3]$
$(2^4\,1; 1^4)$	52	$\begin{pmatrix} 0 & 1 & 1 \\ & 2 & 2 \end{pmatrix}$	$[3^2; \emptyset]$
$(1^3; 3\,2^2\,1^3)$	54	$\begin{pmatrix} 1 & 2 & 2 \\ & 0 & 1 \end{pmatrix}$	$[3\,1; 2]$
$(2^2\,1; 2^2\,1^4)$	54	$\begin{pmatrix} 1 & 1 & 1 \\ & 0 & 3 \end{pmatrix}$	$[4\,1; 1]$
$(1^5; 3\,2^2\,1)$	56	$\begin{pmatrix} 0 & 2 & 2 \\ & 1 & 1 \end{pmatrix}$	$[3; 2\,1]$
$(2^2\,1; 3\,2^2\,1)$	60	$\begin{pmatrix} 2 & 2 \\ & 2 \end{pmatrix}$	$[\emptyset; 2^3]$

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